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CONDITIONS UNDER WHICH THE KERNEL OF A PSEUDOCHARACTER ON A GROUP IS A NORMAL SUBGROUP

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Abstract. We obtain several necessary and sufficient conditions under which the kernel of a pseudocharacter on a group is a normal subgroup of the group.

§ 1. Introduction

In this note, we obtain necessary and sufficient conditions under which the kernel of a pseudocharacter on a group is a normal subgroup of the group. For the generalities concerning pseudocharacters, see [1–4].

§ 2. Preliminaries

Lemma. *Let G be a group, let N be a normal subgroup of G, and let π be the canonical epimorphism of G onto G/N. If a pseudocharacter f on G vanishes on N, then there exists a pseudocharacter φ on the group G/N such that $f = \psi \circ \varphi$. If G is a topological group, N is closed, and f is continuous, then φ is continuous.*

Proof. Let G be a group, let N be a normal subgroup of G, let $g \in G$, $n \in N$, and let f be a pseudocharacter on G vanishing on N. Let $m \in \mathbb{N}$. Then $m|f(gn) - f(g)| = |f(gn)^m) - mf(g)| = |f(g^m(\prod_{k=m-1}^{1} g^{-k}ng^k)n) - f(g^m)| \le c$, since $(\prod_{k=m-1}^{1} g^{-k}ng^k)n \in N$; this implies that $f(gn) = f(g)$. This means that f is constant on every coset of N in G. Define a real-valued

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function φ on G/N by setting $\varphi(gN) = f(g)$ (since f is constant on the cosets of N, it follows that this definition is correct). The above formula for $m|f(gn) - f(g)|$, together with a similar formula for $|f(gn)^{-m}) - f(g)^{-m}|$, shows that φ is a pseudocharacter on G/N, and that $\varphi = \psi \circ \pi$, where π is the canonical epimorphism of G onto G/N. The continuity assertion follows immediately from the last formula.

§ 3. Main Result

Theorem. *Let G be a group, let f be a pseudocharacter on G, and let $N = \ker f$, i.e., $N = \{g \in G : f(g) = 0\}$. The following conditions are equivalent*:

1) N is a normal subgroup of G;

2) N contains the products of its elements, i.e., $f(n_1n_2) = 0$ for every $n_1, n_2 \in N$;

3) $f(gn) = f(g)$ for every $g \in G$ and $n \in N$.

Proof. Recall that N is invariant with respect to all inner automorphisms of the group G (see [1]). Since the restriction of f to every amenable subgroup is an ordinary homomorphism of this subgroup to $\mathbb{R}$ (see [1]), it follows that $f(g^{-1}) = -f(g)$ for every $g \in G$. Therefore, 2) implies 1).

As was established in the proof of the lemma, 1) implies 3).

Since it follows from 3) that $f(n_1n_2) = f(n_1) = 0$ for $n_1, n_2 \in N$, it follows that 3) implies 2).

This completes the proof of the theorem.

§ 4. Discussion

The example of a Guichardet–Wigner pseudocharacter on the universal covering group of the group $\mathrm{SU}(1,1)$, whose kernel is not a normal subgroup because this kernel is nontrivial and the group is simple, shows that the kernel of a pseudocharacter need not be a normal subgroup.

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